INTRO:
DAN CRISTOFARO-GARDINER: FLOER THEORY AND ITS APPLICATIONS

Meditation on

$$\begin{tikzcd} & ECH(Y) \arrow[dl, dash, "\simeq"'] \arrow[dr, dash, "\simeq"] & \\ \widehat{HM}(Y) \arrow[rr, dash, "\simeq"] & & HF^{+}(-Y) \end{tikzcd}$$

Notation: Y closed 3-manifold. Above is a triangle of 3mfld invariants.

ECH = embedded contact homology
$\widehat{HM}$ = Seiberg witten Floer cohomology
HF^+ = Heegaard Floer homology.

$\Big\}$ all vector spaces over $\mathbb{Z}/2\mathbb{Z}$.

Rigged bc. ECH is on top.

ECH is closely connected / sees dynamics.
$\widehat{HM}$ is connected to gauge theory.
HF^+ is especially topological.

User's guide structure of talk.

Application 1: Proof of Weinstein (70s) conjecture (Taubes, 2007)

a) Statement: λ contact form ($\lambda \wedge d\lambda > 0$, so Y oriented). Get a vector field R defined by $d\lambda(R,\cdot) = 0$, $\lambda(R)=1$,
periodic orbits called Reeb orbits (+ Y closed)

Conjecture: $\exists$ a Reeb orbit.

analogues in higher dimensions are wide open.

b) Context: why care?

Hamiltonian mechanics
(M^{2n}, ω), $d\omega = 0$, $\omega^n \neq 0$.
$H: M \to \mathbb{R} \rightsquigarrow X_H$ defined by $\omega(X_H,\cdot) = dH$.

Exercise: X_H exactly encodes Hamilton's equations of motion.

Conservation of energy: flow of X_H preserves H: $X_H(H) = dH(X_H) = \omega(X_H, X_H) = 0$.
Dynamics of X_H preserve H.

What can I say about $H^{-1}(\varepsilon)$? (assuming ε is a regular value)
Must there be periodic orbits along $H^{-1}(\varepsilon)$?
Answer: No :($\exists$ mflds w/ H.v.f. with no periodic orbits.

e.g. Zehnder: Let $M = T^4 = \left(\mathbb{R}/2\pi\mathbb{Z}\right)^4$, $\omega = dx_1 \wedge dx_2 + c\, dx_2 \wedge dx_3 + dx_3 \wedge dx_4$ where $c \in \mathbb{Q}^c$ (2π?)

$H = \sin(x_4)$

Compute $X_H = \cos(x_4)\left(\partial_{x_3} + c\,\partial_{x_1}\right)$. Has no periodic orbits except when $x_4 = 0$, which never happens for a regular value of H.

(Rabinowitz, 70's) $Y = H^{-1}(E)$ (regular value), $M = \mathbb{R}^{2n}$, ω_{std}. Assume that Y is star shaped (transverse to radial v.f $\Sigma x_i dx_i + y_i dy_i$) (H is proper)

$\Rightarrow \exists$ closed orbit.

Fact: Y star shaped is not symplectomorphism invariant. So Weinstein: $Y \subset M$ hypersurface is of contact type if $\exists$ contact form λ on Y s.t $\omega|_Y = d\lambda$.

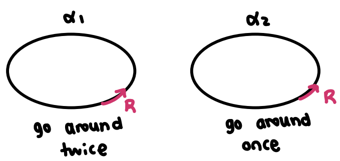
Exercise: $X_H|_Y = f \cdot R$ for some function $f: Y \to \mathbb{R}$.

Upshot: Weinstein's conjecture guarantees periodic orbits of X_H along contact type energy levels.

c) Proof of 3D Weinstein Conjecture

i) More about ECH: it's the homology of a chain complex (ECC, ∂) i.e. ECC is a vector space, and ∂ is a linear map $\partial^2 = 0$. Then $ECH = \ker(\partial)/\mathrm{Im}(\partial)$. ECC is generated by certain finite sets $\{(\alpha_i, m_i)\}$ such that α_i are distinct, embedded Reeb orbits, and $m_i \in \mathbb{Z}_{\geq 1}$.

Generator: e.g. $\{(\alpha_1, 2), (\alpha_2, 1)\}$



by certain, we dont take all

Fact: $\dim(\widehat{HM}(Y)) = \infty$. Due to Kronheimer + Mrowka.

Since $ECH(Y) \cong \widehat{HM}$, $\dim(ECH) = \infty$. But if no Reeb orbits, $\dim(ECH) = 1$ because empty set $\square$

Application 2:

Theorem (C - , Hutchings '2013) $\exists \geq 2$ orbits.

Remark: examples exist with exactly 2, e.g. $H = |z_1|^2 + |z_2|^2$.

Need the following inputs:

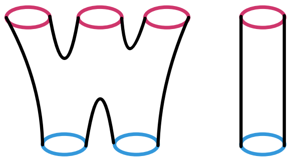
- $\exists\ U: ECH(Y) \to ECH(Y)$ "counts index 2 J-holomorphic curves in $\mathbb{R} \times Y$"
- ∂ counts $I = 1$ curves in $\mathbb{R} \times Y$.

Geometric preliminaries: $(Y,\lambda) \rightsquigarrow \mathbb{R}\times Y$ has an almost complex structure J ($J: TX \to TX$, $J^2=-1$).

J preserves $\ker(\lambda)$, compatible with $d\lambda$

We can study

$u:\underbrace{(\Sigma,j)}_{\text{punctured Riemann surface.}} \to (X,J)$



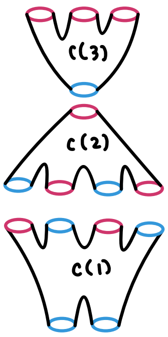
I = "ECH index". Key point: $I(C)=2$, then C is mostly embedded, and has a 2-dim family of deformations.

We can define $U: ECC \to ECC$ by $U(\alpha)=\sum \# \mathcal{M}_z(\alpha,\beta)\beta$, where $\mathcal{M}_z(\alpha,\beta)$ is the space of J-holomorphic curves passing through $z\in\mathbb{R}\times Y$. Notably, U agrees with a similar map on $\widehat{HM}$.

Fact: $U^N \neq 0$ for any N. (via Seiberg-Witten cohomology).

Fact: Since $U^N\neq 0$, for any N, $\exists$ curves $C(1),\dots,C(N)$ each with $C(i)$ index 2 passing through z.

e.g. $U^3\neq 0$



Turns out its very interesting to study $S(N)=\sum_{i=1}^{N}\int_{C(i)} d\lambda$

Fact: (more or less) $\frac{S^2(N)}{N} \longrightarrow \mathrm{Vol}(Y) = \int_Y \lambda\wedge d\lambda$. Weyl law.

Sketch proof of two orbits. Assume $\exists$ only one orbit γ of period T. Can show then $\int_{C(i)} d\lambda > 0$. On the other hand, $\int_{C(i)} d\lambda = mT$ by Stokes' theorem ($m\in\mathbb{Z}_{>0}$). Hence, $S(N)$ grows linearly in N, contradicting the Weyl law. $\square$